MA 441, Fall 2015

Assignment 4.

This homework is due *Thursday*, October 1.

There are total 30 points in this assignment. 27 points is considered 100%. If you go over 27 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.4, 2.5, 3.1 in Bartle–Sherbert.

1. Intervals in $\mathbb{R}$

(1) [3pt] (2.4.19) If u, x, y are any real numbers with $u > 0$ and $x < y$, show that there exists a rational number r such that $x < ur < y$.

Comment. That is, the set $u\mathbb{Q} = \{ru \mid r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.

(2) [4pt] (2.5.10) Let $I_1 = [a_1, b_1] \supseteq I_2 = [a_2, b_2] \supseteq I_3 = [a_3, b_3] \supseteq \dots$ be an infinite nested system of closed intervals. Let $\xi = \sup\{a_n | n \in \mathbb{N}\}$ and $\eta = \inf\{b_n | n \in \mathbb{N}\}$. Prove that

$$[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n.$$

(*Hint:* This is a set equality. Be sure to prove both inclusions.)

2. Limit of a sequence

(3) In this exercise you have to deliver specific inequalities from the definition of the convergent sequence. In each case below, find a number $K \in \mathbb{N}$ such that the corresponding inequality holds for all $n > K$. Give a *specific natural number* as your answer, for example $K = 1000$, or $K = 2 \cdot 10^7$, or $K = 139$, etc. (Not necessarily the smallest possible.)

You can (but you are discouraged to) use a calculator if you want to. However, 1) this problem can be done without using a calculator, 2) even if you do use one, your answers still should easily verifiable without one.

(a) [1pt] $\left|\frac{890534890.6451}{n}\right| < 0.00019011$

(b) [1pt] $\left|\frac{100-n}{n} - (-1)\right| < 0.0054352,$

(c) [2pt] $\left|\frac{200^{10}n+10^{100}}{n^2-10^{200}}\right| < 0.1,$

(d) [2pt] $\left|\frac{\cos(863n)}{\log n}\right| < 0.032432,$

(e) [3pt] (See example 3.1.11(d)) $|\sqrt[n]{n} - 1| < 0.01.$

— see next page —

(4) Reminder. Recall that a sequence $X = (x_n)$ in $\mathbb{R}$ **does not** converge to $x \in \mathbb{R}$ if there is an $\varepsilon_0 > 0$ such that for any $K \in \mathbb{N}$ there is $n_0 > K$ such that following inequality holds: $|x - x_n| \geq \varepsilon_0$.

In each case below find a *real number* $\varepsilon_0 > 0$ that demonstrates that (x_n) does not converge to x.

(a) [2pt] $x_n = 1 + 0.1 \cdot (-1)^{n+1}$, $x = 1$,

(b) [2pt] $x_n = 1/n$, $x = 1/2015$.

(5) (3.1.6cd) Use the definition of limit of a sequence to establish the following limits.

(a) [2pt] $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$.

(b) [2pt] $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

(6) (3.1.8) Let (x_n) be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$.

(a) [2pt] Use definition of limit to prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$.

(b) [2pt] Use definition of limit to prove that if (x_n) converges to x then $(|x_n|)$ converges to $|x|$.

(c) [2pt] Give an example to show that the convergence of $(|x_n|)$ does not imply the convergence of (x_n).